State Space Models Part 3

Periodicity

2 March 2018

Definition of Gaussian Process

- GP is a prior distribution over *function* f.
- Specified by giving
  - mean function $\mu(x)$
  - covariance function $\kappa(x_1, x_2)$
- For any n points $x_1, \ldots, x_n$,

$$\begin{aligned} (y_1, \ldots, y_n)' &\sim \mathrm{Normal}(\boldsymbol{m}, \boldsymbol{C}) \\ y_i &= f(x_i) \\ m_i &= \mu(x_i) \\ C_{ij} &= \kappa(x_i, x_j) \end{aligned}$$

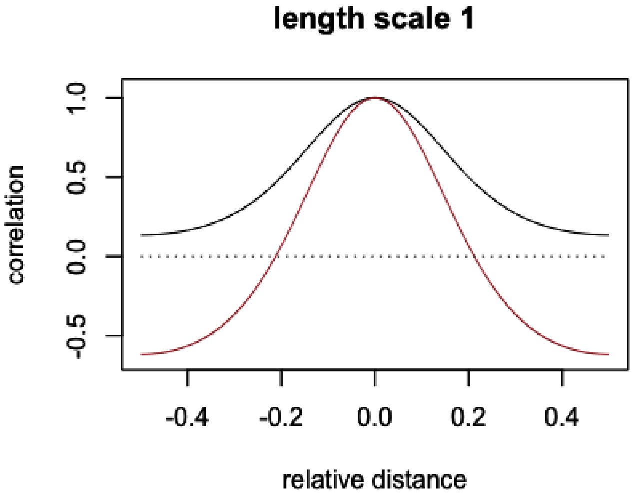
- κ must always produce a valid covariance matrix.

Covariance function / kernels

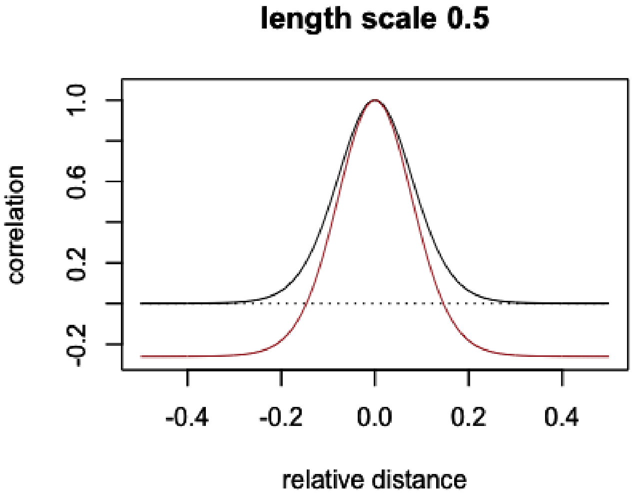
- Symmetric: $\kappa(x_1, x_2) = \kappa(x_2, x_1)$.
- Positive semi-definite (PSD): $v'Cv \geq 0$ for all v
  - $C_{ij} = \kappa(x_i, x_j)$.
  - May be positive definite (PD): $v'Cv > 0$ for all $v \neq \mathbf{0}$.
  - PSD: Cannot choose $f(x_1), \ldots, f(x_n)$ arbitrarily.
- Often *stationary*: $\kappa(x_1, x_2)$ depends only on $x_1 - x_2$
  - In this case, write $\kappa(d)$ where $d = x_1 - x_2$.
- Example: *periodic* kernel

$$\kappa_{\mathrm{P}}(d) = \sigma^2 \exp\left(-\frac{2\sin^2(\pi d/p)}{\ell^2}\right)$$

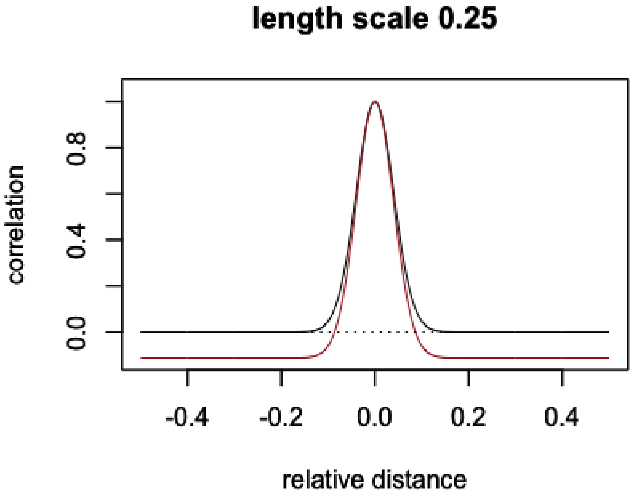
Periodic kernel, $\ell = 1$



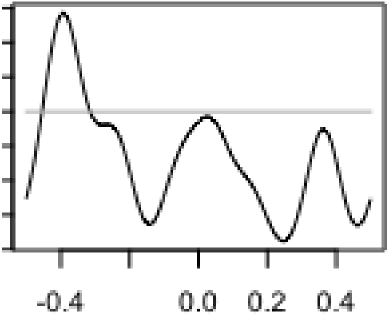
Periodic kernel, $\ell = 0.5$



Periodic kernel, $\ell = 0.25$



Draw from periodic kernel, $\ell = 0.5$

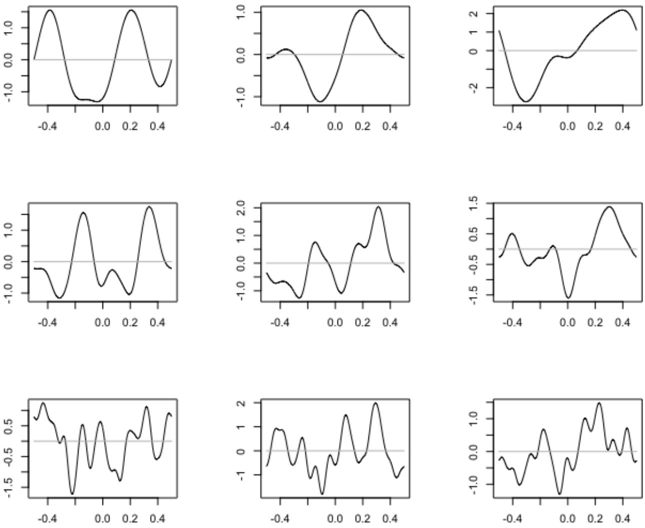


Not centered at zero!

Centering around zero

- Our periodic kernel is positive definite
  - hence cannot impose requirement $\int_0^p f(x)\mathrm{d}x = 0$.
- Modify the kernel:
  - $\kappa^*(d) = s \cdot (\kappa(d) - \Delta)$.
  - Center around zero.
  - Stretch it out so $\kappa(0) = 1$ (or σ^2) again.
- Is it still a valid kernel (positive semi-definite)?
  - Yes.
  - Proof is an exercise for the reader.
- See brown curves in plots of kernel.

Draws from centered periodic kernel



Quasi-periodicity

- Allow small changes from one period to next.
- Product of two valid kernels is a valid kernel.
- Use

$$\kappa(d) = \sigma^2 \kappa_{\mathrm{E}}(d) \kappa_{\mathrm{P}}(d)$$
$$\kappa_{\mathrm{E}}(d) = \phi^d$$
$$\kappa_{\mathrm{P}}(d) = s \cdot \left(\exp\left(-\frac{2\sin^2(\pi d/p)}{\ell^2} \right) - \Delta \right)$$

- Should have ϕ^p very close to 1.
- Δ chosen to center κ_{P}.
- s chosen so that $\kappa_{\mathrm{P}}(0) = 1$.

Review: Quasi-sinusoidal SSM

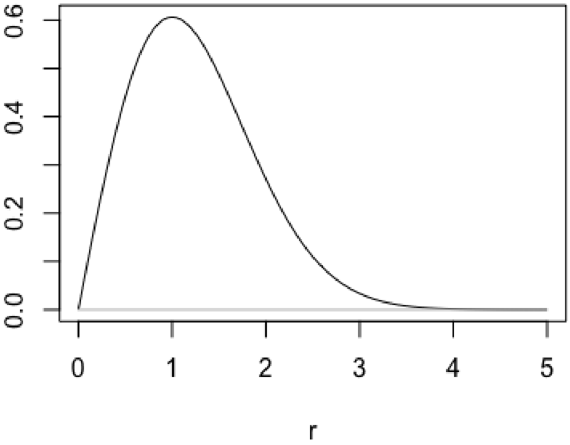
$\mathrm{QS}\left(\theta, \phi, \sigma^2\right)$ is a SSM equivalent to

$$
\begin{aligned}
y_t &= \alpha_{t1} \\
\alpha_1 &\sim \mathrm{Normal}\left(0, \sigma^2 I_2\right) \\
\alpha_{t+1} &= \phi U_\theta \alpha_t + \eta_t \\
\eta_t &\sim \mathrm{Normal}\left(0, \left(1 - \phi^2\right) \sigma^2 I_2\right) \\
U_\theta &= \text{counterclockwise rotation by angle } \theta.
\end{aligned}
$$

If $\phi = 1$ then $y_t = f(t)$, where

$$
\begin{aligned}
a &\sim \mathrm{Rayleigh}\left(\sigma\right) \\
\psi &\sim \mathrm{Uniform}\left(0, 2\pi\right) \\
f(x) &\triangleq a \cos\left(x\theta + \psi\right).
\end{aligned}
$$

Review: Quasi-sinusoidal SSM



Rayleigh distribution for $\sigma = 1$.

Alternative form, quasi-sinusoidal SSM

$\mathrm{QS}\left(\theta, \phi, \sigma^2\right)$ is equivalent to

$$\begin{aligned} y_t &= \alpha_{1t} \cos\left(\theta\left(t-1\right)\right) + \alpha_{2t} \sin\left(\theta(t-1)\right) \\ \alpha_{i,1} &\sim \mathrm{Normal}\left(0, \sigma^2\right) \\ \alpha_{i,t+1} &= \phi \alpha_{i,t} + \eta_{it} \\ \eta_{i,t} &\sim \mathrm{Normal}\left(0, \left(1-\phi^2\right)\sigma^2\right) \end{aligned}$$

Linear regression with time-varying coefficients from AR(1) processes.

$\theta = 0$: $\mathrm{QS}\left(0, \phi, \sigma^2\right)$ yields an AR(1) process for $y_1, y_2, \ldots$

Review: Quasi-periodic SSM

$$\mathrm{QP}\left(p, \phi, \boldsymbol{c}\right) = \mathcal{M}_0 + \mathcal{M}_1 + \cdots + \mathcal{M}_n$$
$$\mathcal{M}_k = \mathrm{QS}\left(2\pi k/p,\ \phi,\ c_k\right)$$

- Set $c_0 = 0$ to center.
- Stationary, mean 0, variance $\sum_{k=0}^{n} c_k$.
- If $\phi = 1$ and p is integer, then $y_t = f(t)$, where

$$a_k \sim \mathrm{Rayleigh}\left(c_k^{1/2}\right), \quad 0 \leq k \leq n$$
$$\psi_k \sim \mathrm{Uniform}\left(0, 2\pi\right), \quad 0 \leq k \leq n$$
$$f(t) = \sum_{k=0}^{n} a_k \cos\left(2\pi k \frac{t}{p} + \psi_k\right).$$

- Whence $\boldsymbol{c}$? Lots of parameters. . .

SSMs and GPs

- SSM: joint distribution for $y_{1:n}$ and $\alpha_{1:n}$ is MV normal.
- Marginalize: distribution for $y_{1:n}$ is MV normal.
- SSMs define GPs on $\mathbb{N}^+ = \{1, 2, \ldots\}$!

- QP $(p, \phi, \boldsymbol{c})$ as a GP:
  - $\mu(t) = 0$
  - $\kappa(t_1, t_2) = C[y_t, y_{t+j}]$ depends only on j:

$$C[y_t, y_{t+j}] = \phi^j \kappa_0(j)$$

$$\kappa_0(j) = \sum_{k=0}^{n} c_k \cos(2\pi kj/p)$$

  - c_k is coefficient of term k in Fourier series for $\kappa_0(\cdot)$!

From kernel to QP SSM coefficients

1. Choose prior variance σ^2.
2. Choose kernel γ: periodic with period 1, $\gamma(0) = 1$.
   - Fully defined by values on $[0, 1/2]$.
3. Find the Fourier series for γ, truncate:

$$\gamma(x) \approx \gamma(x; n) = \sum_{k=0}^{n} a_k \cos(2\pi kx) .$$

   Bochner's Theorem guarantees $a_k \geq 0$.

4. Set $c_k = a_k \sigma^2$ for all k.
5. Then

$$\kappa_0(j) = \sum_{k=0}^{n} a_k \sigma^2 \cos(2\pi kj/p) = \sigma^2 \gamma(j/p; n) .$$

Example

Using periodic kernel:

$$\gamma(x) = \exp\left(-\frac{2\sin^2(\pi x)}{\ell^2}\right)$$
$$= \sum_{k=0}^{\infty} a_k \cos(2\pi kx)$$
$$a_k = b_k \frac{I_k\left(\ell^{-2}\right)}{\exp\left(\ell^{-2}\right)}$$
$$b_k = \begin{cases} 2 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$
$$c_k = a_k \sigma^2.$$

I_k is modified Bessel function of the first kind.

Aliasing (1)

Suppose that p and N are integers.

1. $\mathrm{QS}\left(2\pi(k+Np)/p, \phi, \sigma^2\right) \equiv \mathrm{QS}\left(2\pi k/p, \phi, \sigma^2\right)$ since

$$\frac{2\pi\left(k+Np\right)}{p} = \frac{2\pi k}{p} + 2\pi N.$$

2. $\mathrm{QS}\left(\theta, \phi, \sigma_1^2\right) + \mathrm{QS}\left(\theta, \phi, \sigma_2^2\right) \equiv \mathrm{QS}\left(\theta, \phi, \sigma_1^2 + \sigma_2^2\right)$.
3. So

$$\begin{aligned}
&\mathcal{M}_k + \mathcal{M}_{k+Np} \\
&= \mathrm{QS}\left(2\pi k/p, \phi, c_k\right) + \mathrm{QS}\left(2\pi(k+Np)/p, \phi, c_{k+Np}\right) \\
&\equiv \mathrm{QS}\left(2\pi k/p, \phi, c_k + c_{k+Np}\right)
\end{aligned}$$

4. Can sum all c_{k+Np} into c_k, only keep terms $k < p$.

Aliasing (2)

Suppose that p is an integer.

1. $\mathrm{QS}\left(-\theta, \phi, \sigma^2\right) \equiv \mathrm{QS}\left(\theta, \phi, \sigma^2\right)$ since $\cos\left(-\theta\right) = \cos\left(\theta\right)$.
2. So

$$\begin{aligned}
&\mathcal{M}_k + \mathcal{M}_{p-k} \\
&= \mathrm{QS}\left(2\pi k/p, \phi, c_k\right) + \mathrm{QS}\left(2\pi\left(p-k\right)/p, \phi, c_{p-k}\right) \\
&= \mathrm{QS}\left(2\pi k/p, \phi, c_k\right) + \mathrm{QS}\left(-2\pi k/p, \phi, c_{p-k}\right) \\
&= \mathrm{QS}\left(2\pi k/p, \phi, c_k + c_{p-k}\right).
\end{aligned}$$

Centering

- Started out with $c_0 \neq 0$.
- Unaliasing added to c_0.
- Set $c_0 = 0$ and rescale:

$$c_0^* = 0$$

$$c_k^* = \frac{\sigma^2}{s^2} c_k, \quad 1 \leq k \leq n$$

$$s^2 = \sum_{k=1}^{n} c_k$$

- Yields $\kappa^*(d) = s \cdot (\kappa(d) - c_0)$.

Conclusion

- Can handle multiple seasonality.
- Can have non-integer period.
- Borrow kernel(s) from GP literature.
- Few parameters required, even for long periods:
  - ϕ, σ^2, ℓ.
- Computational cost: $2n$ state vars, $O\left(n^2\right)$.