State Space Models, Lecture 3a

The Formalism

27 April 2018

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Formal Definition of a DLM (1)

Parameters:

- p, the dimensionality of observation vectors.
- m, the dimensionality of the latent state vectors.
- Z_t for all $t \geq 1$, a $p \times m$ matrix relating the latent state to the observations;
- T_t for all $t \geq 1$, an $m \times m$ transition matrix for the latent state;
- H_t for all $t \geq 1$, a $p \times p$ covariance matrix for observation disturbances;
- Q_t for all $t \geq 1$, an $m \times m$ covariance matrix for the latent-state disturbances;
- a_1, an $m \times 1$ vector giving the prior mean for the latent state;
- P_1, an $m \times m$ matrix giving the prior covariance for the latent state.

Formal Definition of a DLM (2)

Variables:

- y_t, the observation vector at time t.
- α_t, the latent state vector at time t.
- ε_t, the observation disturbances at time t.
- η_t, the latent state disturbances at time t.

Equations:

$$
\begin{aligned}
y_t &= Z_t \alpha_t + \varepsilon_t \\
\alpha_1 &\sim \mathrm{Normal}\left(a_1, P_1\right) \\
\alpha_{t+1} &= T_t \alpha_t + \eta_t \\
\varepsilon_t &\sim \mathrm{Normal}\left(\mathbf{0}, H_t\right) \\
\eta_t &\sim \mathrm{Normal}\left(\mathbf{0}, Q_t\right)
\end{aligned}
$$

Example: White Noise

Equation:

$$y_t \sim \mathrm{Normal}\left(\mu, \sigma^2\right)$$

As a DLM:

$$\begin{aligned} Z_t &= 1 & H_t &= \sigma^2 \\ T_t &= 1 & Q_t &= 0 \\ a_1 &= \mu & P_1 &= 0 \end{aligned}$$

Example: Random Walk

Equations:

$$
\begin{aligned}
y_1 &\sim \mathrm{Normal}\left(\mu, \sigma_0^2\right) \\
y_{t+1} &= y_t + \epsilon_t \\
\epsilon_t &\sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right)
\end{aligned}
$$

As a DLM, using $y_t = \alpha_t$:

$$
\begin{aligned}
Z_t &= 1 & H_t &= 0 \\
T_t &= 1 & Q_t &= \sigma_\epsilon^2 \\
a_1 &= \mu & P_1 &= \sigma_0^2
\end{aligned}
$$

Example: Zero-Centered AR(1)

Equations:

$$
\begin{aligned}
y_1 &\sim \mathrm{Normal}\left(0, \sigma^2\right) \\
y_{t+1} &= \phi y_t + \epsilon_t \\
\epsilon_t &\sim \mathrm{Normal}\left(0, \left(1 - \phi^2\right) \sigma^2\right)
\end{aligned}
$$

As a DLM, with $\alpha_t = y_t$:

$$
\begin{aligned}
Z_t &= 1 & H_t &= 0 \\
T_t &= \phi & Q_t &= \left(1 - \phi^2\right) \sigma^2 \\
a_1 &= 0 & P_1 &= \sigma^2
\end{aligned}
$$

Example: General AR(1)

Equations:

$$
\begin{aligned}
y_1 &\sim \mathrm{Normal}\left(\mu, \sigma^2\right) \\
y_{t+1} &= \mu + \phi\left(y_t - \mu\right) + \epsilon_t \\
\epsilon_t &\sim \mathrm{Normal}\left(0, \left(1-\phi^2\right)\sigma^2\right)
\end{aligned}
$$

As a DLM, with $\alpha_{t1} = y_t - \mu$ and $\alpha_{t2} = \mu$:

$$
\begin{aligned}
Z_t &= (1, 1) & H_t &= 0 \\
T_t &= \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} & Q_t &= \begin{pmatrix} \left(1-\phi^2\right)\sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \\
a_1 &= (0, \mu) & P_1 &= \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

Example: Local Linear Trend (RW)

Equations:

$$
\begin{aligned}
y_t &= \alpha_t + \varepsilon_t & \varepsilon_t &\sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right) \\
\alpha_{t+1} &= \alpha_t + \beta_t + \eta_{\alpha t} & \eta_{\alpha t} &\sim \mathrm{Normal}\left(0, \sigma_{\eta\alpha}^2\right) \\
\beta_{t+1} &= \beta_t + \eta_{\beta t} & \eta_{\beta t} &\sim \mathrm{Normal}\left(0, \sigma_{\eta\beta}^2\right) \\
\alpha_1 &\sim \mathrm{Normal}\left(\mu_\alpha, \sigma_\alpha^2\right) & \beta_1 &\sim \mathrm{Normal}\left(\mu_\beta, \sigma_\beta^2\right)
\end{aligned}
$$

As a DLM:

$$
\begin{aligned}
Z_t &= (1, 0) & H_t &= \sigma_\epsilon^2 \\
T_t &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & Q_t &= \begin{pmatrix} \sigma_{\eta\alpha}^2 & 0 \\ 0 & \sigma_{\eta\beta}^2 \end{pmatrix} \\
a_1 &= (\mu_\alpha, \mu_\beta) & P_1 &= \begin{pmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_\beta^2 \end{pmatrix}
\end{aligned}
$$

Example: Quasi-Sinusoidal

Equations:

$$
\begin{aligned}
y_t &= \alpha_{t1} \\
\alpha_1 &\sim \mathrm{Normal}\left(0, \boldsymbol{\Sigma}\right) \\
\alpha_{t+1} &= \phi U_\theta \alpha_t + \eta_t \\
\eta_t &\sim \mathrm{Normal}\left(0, \left(1 - \phi^2\right) \boldsymbol{\Sigma}\right) \\
\boldsymbol{\Sigma} &= \mathrm{diag}\left(\sigma^2, \sigma^2\right) \\
U_\theta &= \text{counterclockwise rotation by angle } \theta
\end{aligned}
$$

As a DLM:

$$
\begin{aligned}
Z_t &= (1, 0) & H_t &= 0 \\
T_t &= \phi U_\theta & Q_t &= \left(1 - \phi^2\right) \boldsymbol{\Sigma} \\
a_1 &= (0, 0) & P_t &= \boldsymbol{\Sigma}
\end{aligned}
$$

Example: Quasi-Periodic for moderate P

Equations

$$
\begin{aligned}
y_t &= \beta_{t,1} \\
\beta_1 &\sim \text{Normal}\left(0, \boldsymbol{\Sigma}_{\text{eff}}\right) \\
\beta_{t+1} &= \phi \cdot \left(\left(-\sum_{i=1}^{N-1} \beta_{t,i} \right), \beta_{t,1}, \ldots, \beta_{t,N-2} \right)' + \epsilon_t \\
\epsilon_t &\sim \text{Normal}\left(\mathbf{0}, \rho \boldsymbol{\Sigma}_{\text{eff}}\right)
\end{aligned}
$$

As a DLM:

$$
Z_t = (1, 0, \ldots, 0) \qquad H_t = 0
$$

$$
T_t = \phi \begin{pmatrix} -1 & \cdots & -1 & -1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \qquad Q_t = \rho \boldsymbol{\Sigma}_{\text{eff}}
$$

$$
a_1 = (0, 0) \qquad P_t = \boldsymbol{\Sigma}_{\text{eff}}
$$