State Space Models, Lecture 2

Local Linear Trend, Regression, Periodicity

09 April 2018

Review of Lecture 1

State-space models:

- Unobserved hidden state
- Observed values: function of hidden state, plus noise
- Sum of SSMs

DLMs:

- Linear evolution of state; Gaussian (normal) noise
- Random walk: σ_η^2
- AR(1): σ_η^2, ϕ, μ
- Local level model: RW or AR1 plus noise
- Integrated RW / AR(1)

Local Linear Trend

Integrated RW / AR(1), plus LLM.

$$\beta \sim \text{AR1}\left(\phi_\beta, \mu_\beta, \sigma_\beta^2\right)$$

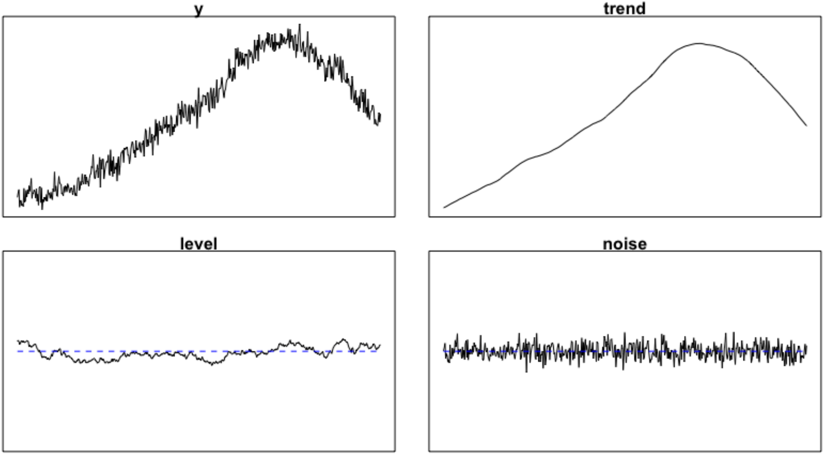
$$\begin{aligned} \gamma_1 &\sim \mathrm{Normal}\left(\mu_{\gamma 0}, \sigma_{\gamma 0}^2\right) \\ \gamma_{t+1} &= \gamma_t + \beta_t \end{aligned}$$

$$\alpha \sim \text{AR1}\left(\phi_\alpha, 0, \sigma_\alpha^2\right)$$

$$\begin{aligned} y_t &= \gamma_t + \alpha_t + \epsilon_t \\ \epsilon_t &\sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right) \end{aligned}$$

γ_t: trend; α_t: local level; ϵ_t: noise.

Local Linear Trend (plot)



Time-varying Linear Regression

Linear regression with coefficients that vary over time.

$$
\begin{aligned}
y_t &= X_t \alpha_t + \epsilon_t \\
\epsilon_t &\sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right) \\
\alpha &\sim \mathrm{AR1}\left(\phi, \mu_\alpha, \sigma_\alpha^2\right)
\end{aligned}
$$

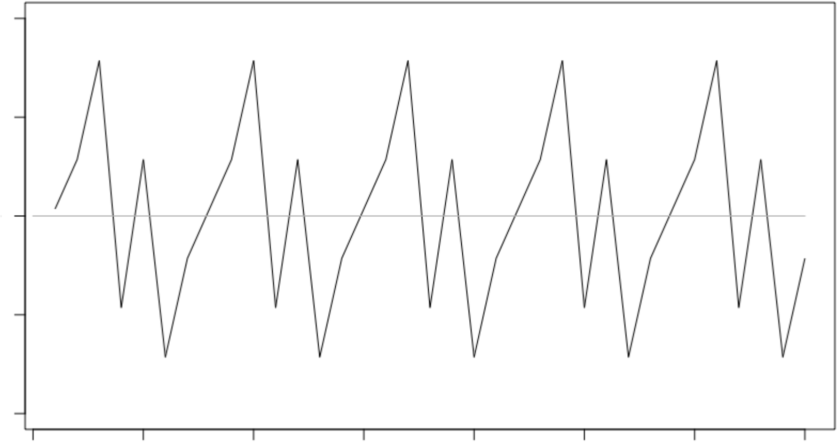
(Note: X_t is a row vector.)

Use case: external covariates.

Seasonality / Periodicity

"Seasonality": repeating periodic pattern

- Daily
- Weekly
- Yearly



Periodic Model – Attempt 1

If period is N, use

$$\begin{aligned} y_t &= \alpha_t \\ \alpha_1 &\sim \text{Normal}\left(0, \sigma^2\right) \\ &\vdots \\ \alpha_N &\sim \text{Normal}\left(0, \sigma^2\right) \\ \alpha_t &= \alpha_{t-N} \quad \text{for } t > N \end{aligned}$$

But this is non-Markovian:

- Should only have prior on α_1.
- α_t should depend only on α_{t-1}.

Periodic Model – Attempt 2

Use standard trick:

Let $\beta_t = (\alpha_t, \alpha_{t-1}, \ldots, \alpha_{t-N+1})'$.

Then

$$\begin{aligned} y_t &= \beta_{t,1} \\ \beta_{1,i} &\sim \mathrm{Normal}\left(0, \sigma^2\right), \quad 1 \le i \le N \\ \beta_{t+1} &= (\beta_{t,N}, \beta_{t,1}, \ldots, \beta_{t,N-1})' \end{aligned}$$

But... not zero-centered: we require

$$\sum_{i=1}^{N} \beta_{t,i} = 0.$$

Periodic Model – Attempt 3

Define

$$\beta_{t,N} = -\sum_{i=1}^{N-1} \beta_{t,i}.$$

Then

$$\begin{aligned} y_t &= \beta_{t,1} \\ \beta_{1,i} &\sim \mathrm{Normal}\left(0, \sigma^2\right), \quad 1 \leq i \leq N-1 \\ \beta_{t+1} &= \left(-\sum_{i=1}^{N-1} \beta_{t,i},\, \beta_{t,1}, \ldots, \beta_{t,N-2}\right)' \end{aligned}$$

But the prior is asymmetric:

$$\beta_{1,N} \sim \mathrm{Normal}\left(0, (N-1)\,\sigma^2\right)$$

Multivariate Normal Distribution

$$\boldsymbol{x} \sim \mathrm{MVNormal}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$$

N correlated variables, each having a normal distribution:

- μ_i: mean for x_i
- Σ_{ii}: variance for x_i
  - $\sigma_i = \Sigma_{ii}^{1/2}$
- Σ_{ij}: covariance for x_i and x_j
  - correlation is $\Sigma_{ij} / \left(\sigma_i \sigma_j\right)$.

Symmetric Effects Prior

If $\boldsymbol{x}$ has length $N - 1$ and we use

$$\boldsymbol{x} \sim \mathrm{MVNormal}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\mathrm{eff}}\right)$$

$$\boldsymbol{\Sigma}_{\mathrm{eff}} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 & \rho\sigma^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

$$\rho = -1/\left(N - 1\right)$$

$$x_N = -\sum_{i=1}^{N-1} x_i$$

then

- mean of x_i is 0, $1 \leq i \leq N$;
- variance of x_i is σ^2, $1 \leq i \leq N$.

Periodic Model – Attempt 4

Define

$$
\begin{aligned}
y_t &= \beta_{t,1} \\
\beta_1 &\sim \mathrm{MVNormal}\left(0, \boldsymbol{\Sigma}_{\mathrm{eff}}\right) \\
\beta_{t+1} &= \left(-\sum_{i=1}^{N-1} \beta_{t,i},\ \beta_{t,1}, \ldots, \beta_{t,N-2}\right)'
\end{aligned}
$$

But what if we want to allow the periodic pattern to slowly change over time?

Quasi-Periodic Model – Attempt 1

Add some random drift:

$$\begin{aligned} y_t &= \beta_{t,1} \\ \beta_1 &\sim \text{MVNormal}\left(0, \mathbf{\Sigma}_{\text{eff}}\right) \\ \beta_{t+1} &= \left(-\sum_{i=1}^{N-1} \beta_{t,i},\, \beta_{t,1}, \ldots, \beta_{t,N-2} \right)' + \epsilon_t \\ \epsilon_t &\sim \text{MVNormal}\left(\mathbf{0}, \rho\mathbf{\Sigma}_{\text{eff}}\right) \end{aligned}$$

where $N\rho \ll 1$.

But random-walk behavior:

$$V\left[\beta_{ti}\right] = \sigma^2 \left(1 + (t-1)\rho\right), \quad t \geq 1,\ 1 \leq i \leq N$$

Magnitude of pattern increases, on average, over time.

Quasi-Periodic Model – Attempt 2

Add some damping (like AR(1) model):

$$
\begin{aligned}
y_t &= \beta_{t,1} \\
\beta_1 &\sim \mathrm{MVNormal}\left(0, \boldsymbol{\Sigma}_{\mathrm{eff}}\right) \\
\beta_{t+1} &= \phi \cdot \left(-\sum_{i=1}^{N-1} \beta_{t,i},\, \beta_{t,1}, \ldots, \beta_{t,N-2} \right)' + \epsilon_t \\
\epsilon_t &\sim \mathrm{MVNormal}\left(\mathbf{0}, \rho \boldsymbol{\Sigma}_{\mathrm{eff}}\right) \\
\rho &= 1 - \phi^2
\end{aligned}
$$

where $N\rho \ll 1$. Guarantees

$$
V\left[\beta_{t,i}\right] = \sigma^2, \quad t \geq 1,\ 1 \leq i \leq N
$$

But... What if N is large? (complexity, estimation)
Non-integer periods?

Fourier Series

Decompose periodic function $f(x)$:

$$f(x) = \sum_{k=1}^{\infty} \left(a_k \sin\left(2\pi kx/P\right) + b_k \cos\left(2\pi kx/P\right) \right)$$

where P is the period.

- $a_k \to 0$, $b_k \to 0$ as $k \to \infty$
- smoother functions have fewer large a_k, b_k values
- approximate $f(x)$ by truncating series.

Equivalently, use $a_k \sin\left(2\pi kx/P + \varphi_k\right)$.

Quasi-Sinusoidal

Define $\mathrm{QS}\left(\theta, \phi, \sigma^2\right)$

$$
\begin{aligned}
y_t &= \alpha_{t1} \\
\alpha_1 &\sim \mathrm{Normal}\left(0, \boldsymbol{\Sigma}\right) \\
\alpha_{t+1} &= \phi U_\theta \alpha_t + \eta_t \\
\eta_t &\sim \mathrm{Normal}\left(0, \left(1 - \phi^2\right) \boldsymbol{\Sigma}\right) \\
\boldsymbol{\Sigma} &= \mathrm{diag}\left(\sigma^2, \sigma^2\right) \\
U_\theta &= \text{counterclockwise rotation by angle } \theta \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned}
$$

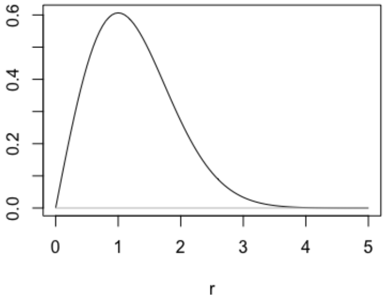
ϕ and η_t give us the "quasi."

Quasi-Sinusoidal (2)

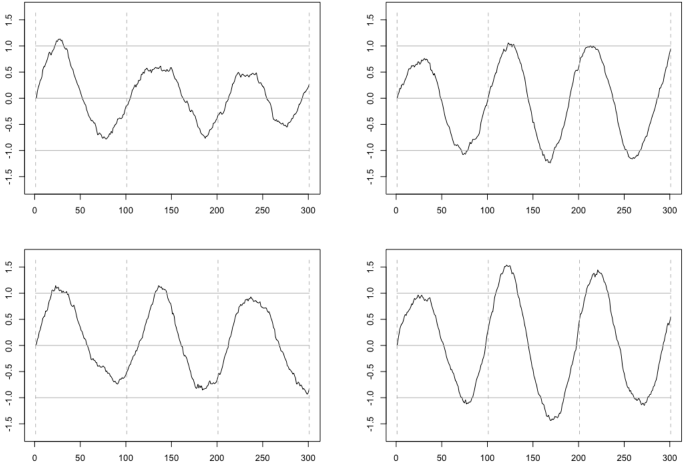
Some notes:

- Period of L corresponds to $\theta = 2\pi/L$.
- To be approximately sinusoidal, ϕ^L should be close to 1.
- If $\phi = 1$ then $y_t = f(t)$, where

$$a \sim \mathrm{Rayleigh}(\sigma)$$
$$\psi \sim \mathrm{Uniform}(0, 2\pi)$$
$$f(x) \triangleq a \cos(x\theta + \psi).$$



Plots: $L = 100$, $\phi^L = 0.95$, $\sigma = 1$



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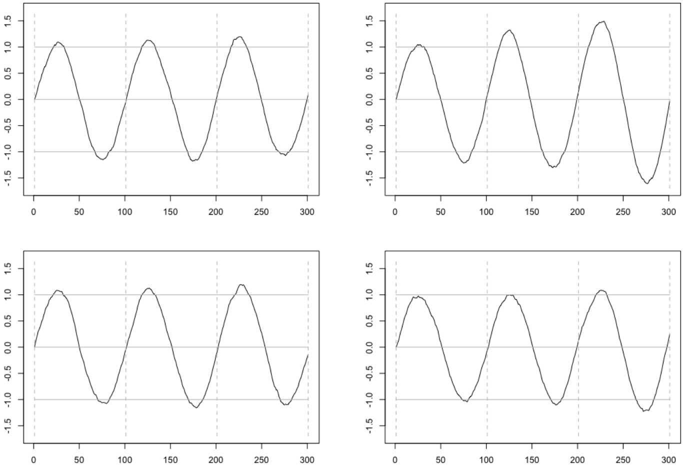

Quasi-Periodic Model – Attempt 3

$$\mathrm{QP}\,(L, \phi, c) = \mathcal{M}_1 + \cdots + \mathcal{M}_n$$
$$\mathcal{M}_k = \mathrm{QS}\left(2\pi k/L,\ \phi,\ c_k^2\sigma^2\right)$$
$$\sum_{k=1}^{n} c_k^2 = 1.$$

Notes:

- Stationary mean is 0.
- Stationary variance is σ^2.
- Non-integer periods allowed.
- Smaller n / more rapidly decreasing c_k mean smoother pattern.

But how do we choose the coefficients c_k?