State Space Models, Lecture 1

Introduction to SSMs

26 March 2018

Why State-Space Models?

Flexible time-series analysis:

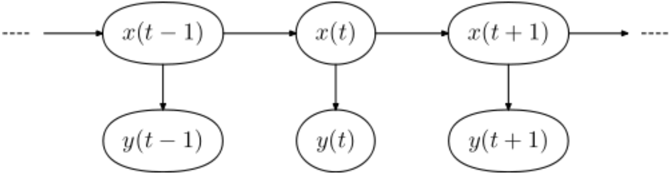
- Forecasting
  - Predictive distribution / interval
- Anomaly detection
- Multiple related time series
- Decomposition:
  - local level
  - local linear trend
  - periodicity
  - shocks
  - noise
- Smoothing
- Missing data
- Time-varying linear regression / factor analysis

Time Series Models

- Joint probability distribution over random variables $y_1, y_2, \ldots, y_t, \ldots$.
- Variable y_i may be scalars or vectors.
- Discrete y_i:
  $\Pr(y_1 = Y_1, \ldots, y_t = Y_t \mid \theta)$
- Continuous y_i:
  $\Pr(Y_1 - \epsilon \leq y_1 \leq Y_1 + \epsilon, \ldots, Y_t - \epsilon \leq y_t \leq Y_t + \epsilon \mid \theta)$
- Derive algorithms from model:
  - Estimation of parameters
  - Forecasting of future y_i (or sums)
    - with prediction intervals / quantiles
  - Decomposition (local level & linear trend, periodicity, shocks)

A Model Hierarchy

- General time-series models:
  - arbitrary joint distribution
- State-space models:
  - hidden state sequence $\alpha_1, \alpha_2, \ldots$ with Markov property
  - observation distribution for y_t conditioned on state α_t.



- Dynamic (generalized) linear models
  - α_{t+1} is linear fct of α_t, plus Gaussian noise
  - y_t (or $f(y_t)$) has distribution with mean μ_t that is linear fct of α_t.
    - $y_t \sim \text{Normal}\left(\mu_t, \sigma^2\right)$ for DLM.
    - $y_t \sim \text{Poisson}\left(\exp\left(\mu_t\right)\right)$ for dynamic Poisson (counts).

Summing DLMs

Suppose that

- $u_1, u_2, \ldots, u_t, \ldots \sim \mathrm{DLM}_u$
- $v_1, v_2, \ldots, v_t, \ldots \sim \mathrm{DLM}_v$
- $y_t = u_t + v_t$

Then

- $y_1, y_2, \ldots, y_t, \ldots \sim \mathrm{DLM}_y$

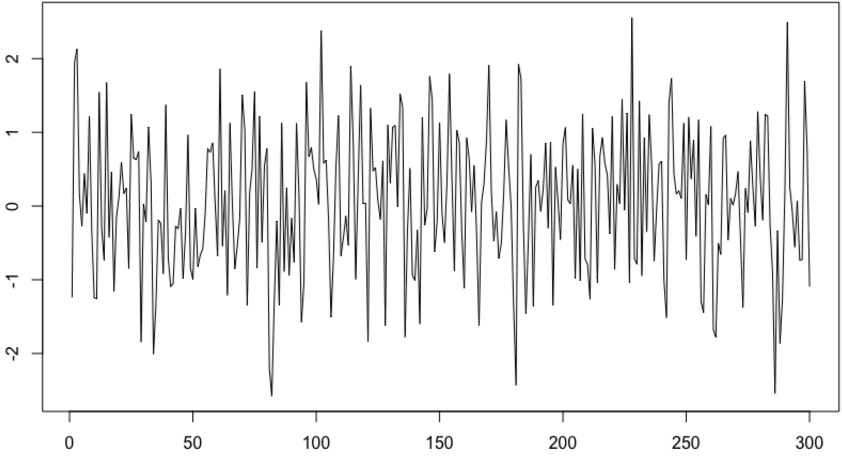
Similarly for generalized DLM.

Basis for decomposition—sum up

- DLM for local linear trend
- DLM for "random walk" deviations from trend
- DLM for daily/weekly patterns
- DLM for effects of external events

White Noise

$$y_t \sim \text{Normal}\left(\mu, \sigma^2\right)$$
$$\theta = \left(\mu, \sigma^2\right)$$

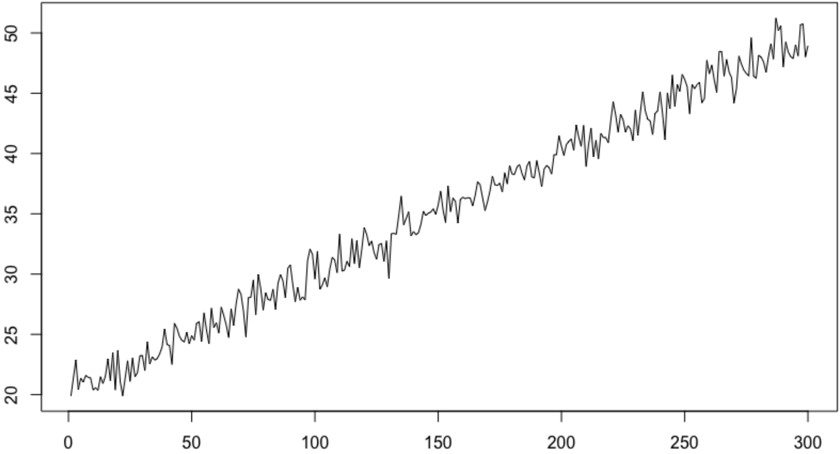


Linear Regression

$$y_t = \beta t + \gamma + \epsilon_t$$

$$\epsilon_t \sim \mathrm{Normal}\left(0, \sigma^2\right)$$

$$\theta = \left(\beta, \gamma, \sigma^2\right)$$



Linear Regression (Alternate Form)

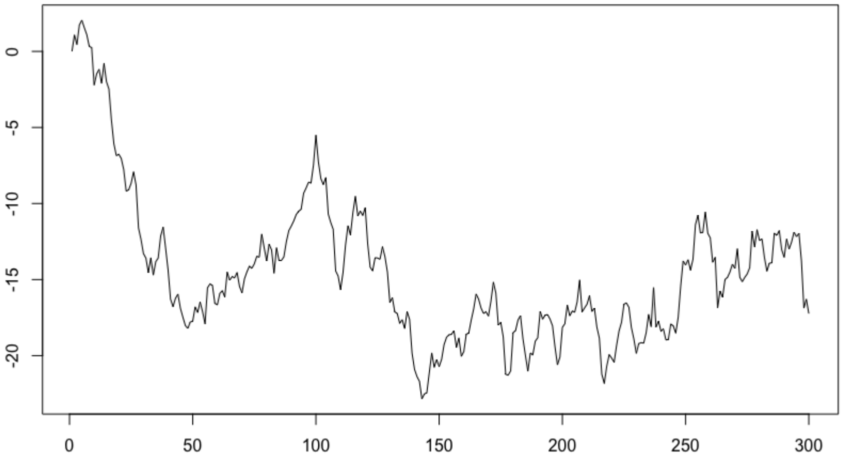
$$\alpha_1 = \beta + \gamma$$
$$\alpha_{t+1} = \alpha_t + \beta$$
$$y_t = \alpha_t + \epsilon_t$$
$$\epsilon_t \sim \mathrm{Normal}\left(0, \sigma^2\right)$$

Suggests generalizations:

- $\alpha_1 \sim \mathrm{Normal}\left(\mu, \sigma_0^2\right)$
- Let β vary slowly over time.

Random Walk

$$
\begin{aligned}
y_1 &\sim \text{Normal}\left(\mu, \sigma_0^2\right) \\
y_{t+1} &= y_t + \epsilon_t \\
\epsilon_t &\sim \text{Normal}\left(0, \sigma_\epsilon^2\right) \\
\theta &= \left(\mu, \sigma_0^2, \sigma_\epsilon^2\right)
\end{aligned}
$$

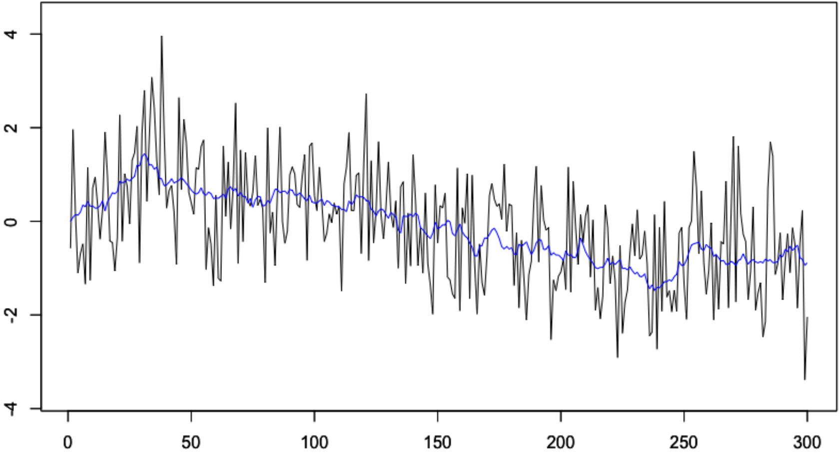


Local Level Model

Random walk + white noise.

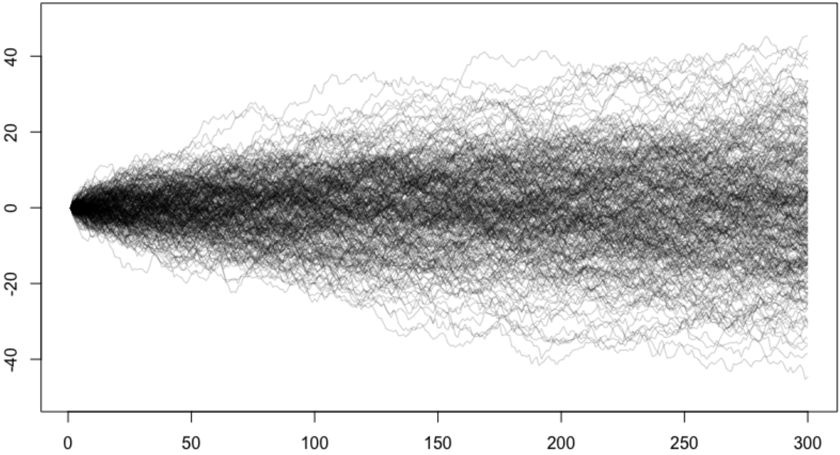
$$
\begin{aligned}
\alpha_1 &\sim \text{Normal}\left(\mu, \sigma_0^2\right) \\
\alpha_{t+1} &= \alpha_t + \eta_t \\
\eta_t &\sim \text{Normal}\left(0, \sigma_\eta^2\right) \\
y_t &= \alpha_t + \epsilon_t \\
\epsilon_t &\sim \text{Normal}\left(0, \sigma_\epsilon^2\right) \\
\theta &= \left(\mu, \sigma_0^2, \sigma_\eta^2, \sigma_\epsilon^2\right)
\end{aligned}
$$

Local Level Model (plot)



$\sigma_\eta = 0.1, \ \sigma_\epsilon = 1$

Random Walks Can Wander Far



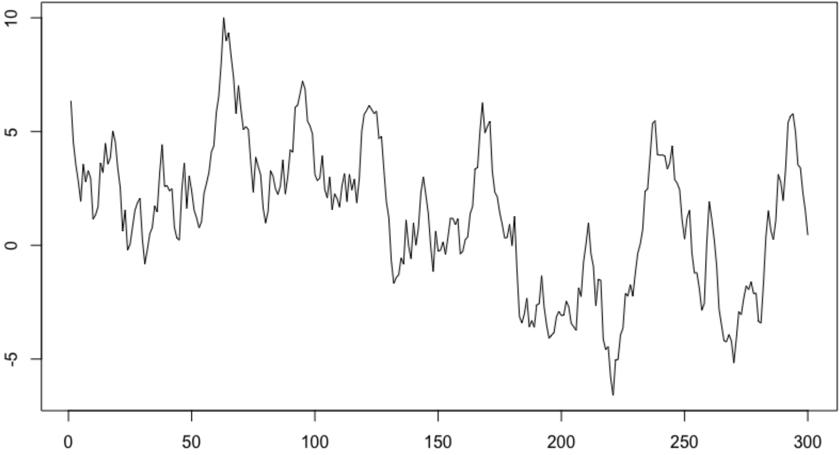
Square-root growth.

AR(1)

Mostly stay within $\mu \pm \sigma_0$.
Random-walk behavior with variance σ_{rw}^2.

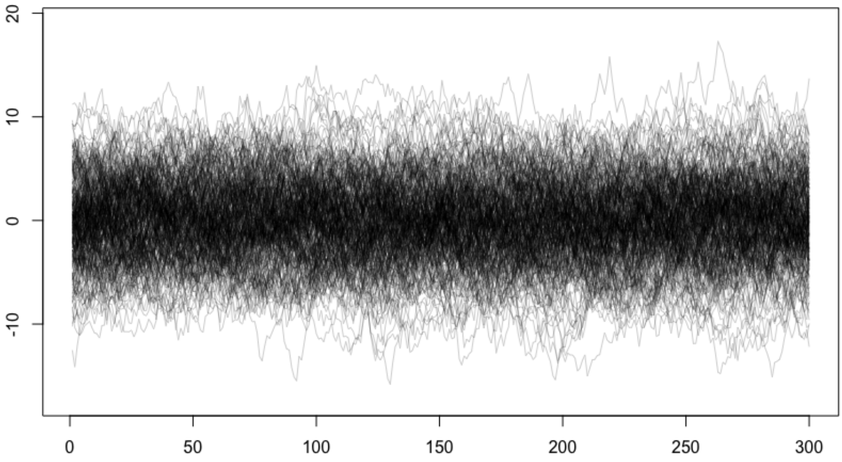
$$
\begin{aligned}
y_1 &\sim \mathrm{Normal}\left(\mu, \sigma_0^2\right) \\
y_{t+1} &= \mu + \phi\left(y_t - \mu\right) + \epsilon_t \\
\epsilon_t &\sim \mathrm{Normal}\left(0, \sigma_{\mathrm{rw}}^2\right) \\
\phi &= \sqrt{1 - \frac{\sigma_0^2}{\sigma_{\mathrm{rw}^2}}}
\end{aligned}
$$

AR(1) example plot



$\sigma_{\mathrm{rw}=1}$, $\sigma_0 = 4$, $\mu = 0$

AR(1) example plot repeated



Integrated Random Walk

Let $y_{t+1} - y_t$ be a random walk:

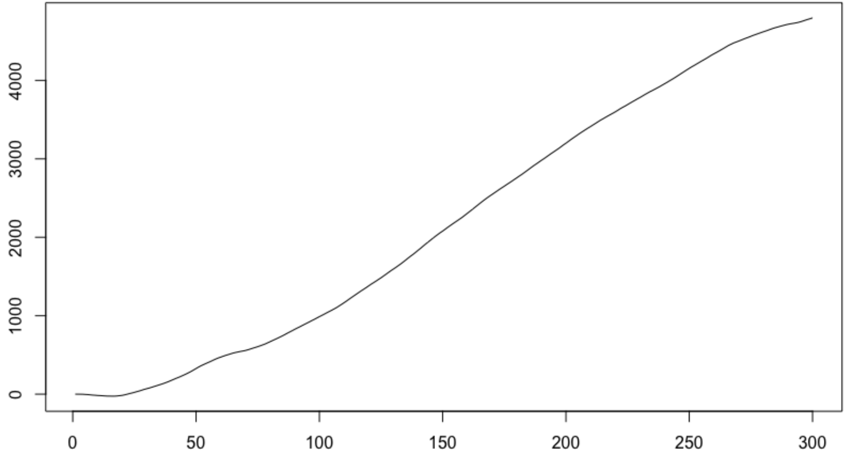
$$\begin{aligned} y_1 &\sim \text{Normal}\left(\mu_{y0}, \sigma^2_{y0}\right) \\ y_{t+1} &= y_t + \beta_t \end{aligned}$$

and

$$\begin{aligned} \beta_1 &\sim \text{Normal}\left(\mu_{\beta 0}, \sigma^2_{\beta 0}\right) \\ \beta_{t+1} &\sim \beta_t + \epsilon_t \\ \epsilon_t &\sim \text{Normal}\left(0, \sigma^2_{\epsilon}\right) \end{aligned}$$

Compare to alternate form of lin regr.

Integrated Random Walk (plot)



Integrated AR(1)

Let $y_{t+1} - y_t$ be a random walk:

$$\begin{aligned} y_1 &\sim \text{Normal}\left(\mu_{y0}, \sigma^2_{y0}\right) \\ y_{t+1} &= y_t + \beta_t \end{aligned}$$

and

$$\begin{aligned} \beta_1 &\sim \text{Normal}\left(\mu_{\beta 0}, \sigma^2_{\beta 0}\right) \\ \beta_{t+1} &\sim \mu_{\beta 0} + \phi\left(\beta_t - \mu_{\beta 0}\right) + \epsilon_t \\ \epsilon_t &\sim \text{Normal}\left(0, \sigma^2_{\epsilon}\right) \end{aligned}$$

Integrated AR(1) (plot)